Introduction to Logic (CS & MA) 2015

Answers Resit Exam of 26 November

1. (a) $\neg(R \vee S) \to C$
   C: Miko will come.
   R: It rains.
   S: It snows.

   (b) $L \wedge (D \to \neg A)$
   L: Sam likes coffee.
   D: Sam drinks coffee.
   A: Sam is fully awake.

2. Translation key:

   a: Amir   m(x): the mentor of x   K(x,y): x knows y

   (a) $\neg\exists x(x = m(m(x)))$

   (b) $\forall x(m(m(x)) = a \to \exists y(K(x, y) \wedge \neg\exists z(y = m(z))))$

3. (a)

A	B	C	$(A$	$\leftrightarrow$	$B)$	$\leftrightarrow$	$((C$	$\leftrightarrow$	$\neg A)$	$\leftrightarrow$	$(B$	$\leftrightarrow$	$C))$
T	T	T	T	T	T	F	T	F	F	F	T	T	T
T	T	F	T	T	T	F	F	T	F	F	T	F	F
T	F	T	T	F	F	F	T	F	F	T	F	F	T
T	F	F	T	F	F	F	F	T	F	T	F	T	F
F	T	T	F	F	T	F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F	F	T	T	T	F	F
F	F	T	F	T	F	F	T	T	T	F	F	F	T
F	F	F	F	T	F	F	F	F	T	F	F	T	F
1	2	3	1	5	2	9	3	6	4	8	2	7	3

The numbers in the last row indicate the order in which the columns are computed. The final column (numbered 9) only contains the value F, so the sentence is *not satisfiable.*

(b)

A	B	C	$(\neg A$	$\to$	$B)$	$\wedge$	C	$\Leftrightarrow$	$\neg$	$(A$	$\wedge$	$C)$	$\to$	B
T	T	T	F	T	T	T	T	T	F	T	T	T	T	T
T	T	F	F	T	T	F	F	F	T	T	F	F	T	T
T	F	T	F	T	F	T	T	T	F	T	T	T	T	F
T	F	F	F	T	F	F	F	T	T	T	F	F	F	F
F	T	T	T	T	T	T	T	T	T	F	F	T	T	T
F	T	F	T	T	T	F	F	F	T	F	F	F	T	T
F	F	T	T	F	F	F	T	T	T	F	F	T	F	F
F	F	F	T	F	F	F	F	T	T	F	F	F	F	F
1	2	3	4	5	2	6	3	10	8	1	7	3	9	2

The numbers in the last row indicate the order in which the columns are computed. The final column (numbered 10) does not only contain the value T, so the sentences are *not tautologically equivalent.*

4. (a)

1. $A \to (B \lor C)$
2. $\neg B$
3. $\neg C$
4. A
5. $B \lor C$ — $\to$ Elim: 1, 4
6. B
7. $\bot$ — $\bot$ Intro: 6, 2
8. C
9. $\bot$ — $\bot$ Intro: 8, 3
10. $\bot$ — $\lor$ Elim: 5, 6–7, 8–9
11. $\neg A$ — $\neg$ Intro: 4–10
12. $\neg C \to \neg A$ — $\to$ Intro: 3–11
13. $\neg B \to (\neg C \to \neg A)$ — $\to$ Intro: 2–12

(b)

1. $(A \leftrightarrow B) \leftrightarrow C$
2. A
3. B
4. A
5. B — Reit: 3
6. B
7. A — Reit: 4
8. $A \leftrightarrow B$ — $\leftrightarrow$ Intro: 4–5, 6–7
9. C — $\leftrightarrow$ Elim: 1, 8
10. $B \to C$ — $\to$ Intro: 3–9
11. $A \to (B \to C)$ — $\to$ Intro: 2–10

(c)

1. $\exists x(a = x \land x = b)$
2. $\boxed{c}$
3. $c = a$
4. $\boxed{d}\ a = d \land d = b$
5. $a = d$ — $\land$ Elim: 4
6. $c = d$ — $=$ Elim: 3, 5
7. $d = b$ — $\land$ Elim: 4
8. $c = b$ — $=$ Elim: 6, 7
9. $c = b$ — $\exists$ Elim: 1, 4–8
10. $c = a \to c = b$ — $\to$ Intro: 3–9
11. $\forall x(x = a \to x = b)$ — $\forall$ Intro: 2–10

(d)

1. $\neg\exists x(P(x) \wedge \neg Q(x))$	
2. $\boxed{a}$	
3. $P(a)$	
4. $\neg Q(a)$	
5. $P(a) \wedge \neg Q(a)$	$\wedge$ Intro: 3, 4
6. $\exists x(P(x) \wedge \neg Q(x))$	$\exists$ Intro: 5
7. $\bot$	$\bot$ Intro: 6, 1
8. $\neg\neg Q(a)$	$\neg$ Intro: 4–7
9. $Q(a)$	$\neg$ Elim: 8
10. $P(a) \to Q(a)$	$\to$ Intro: 3–9
11. $\forall x(P(x) \to Q(x))$	$\forall$ Intro: 2–10

5. (a) $\forall x\forall y((\mathsf{Cube}(x) \wedge \mathsf{Cube}(y) \wedge \mathsf{SameRow}(x, y)) \to x = y)$

   (b) The only objects x that satisfy $\mathsf{Tet}(x)$ are b and d. We claim that there are objects y and z such that both $\mathsf{SameCol}(b, y) \wedge \mathsf{SameRow}(y, z) \wedge \mathsf{Cube}(z)$ and $\mathsf{SameCol}(d, y) \wedge \mathsf{SameRow}(y, z) \wedge \mathsf{Cube}(z)$ hold. To show that, we take d for y, and e for z. It is easily verified that $\mathsf{SameCol}(b, d)$, $\mathsf{SameCol}(d, d)$, $\mathsf{SameRow}(d, e)$ and $\mathsf{Cube}(e)$ all hold. So the sentence is true.

   (c) We shall make the sentence false by making the instance $\forall y(\mathsf{SameCol}(d, y) \to d = y) \to \mathsf{Cube}(d)$ false. Since the conclusion $\mathsf{Cube}(d)$ is false, it suffices to make the premiss $\forall y(\mathsf{SameCol}(d, y) \to d = y)$ true. This is realised by taking object b away, for then there are no two objects in the same column.
   Analogously, the sentence becomes false when we remove object d.

6. (a) $\mathfrak{M} \models R(x, z) \wedge (P(a) \vee Q(b))[h]$

$\Leftrightarrow$ { definition of satisfaction for conjunction }

$\mathfrak{M} \models R(x, z)[h]$ and $\mathfrak{M} \models P(a) \vee Q(b)[h]$

$\Leftrightarrow$ { definition of satisfaction for disjunction }

$\mathfrak{M} \models R(x, z)[h]$ and ($\mathfrak{M} \models P(a)[h]$ or $\mathfrak{M} \models Q(b)[h]$)

$\Leftrightarrow$ { definition of satisfaction for atomic formulae }

$\langle [\![x]\!]^{\mathfrak{M}}_h, [\![z]\!]^{\mathfrak{M}}_h \rangle \in \mathfrak{M}(R)$ and ($[\![a]\!]^{\mathfrak{M}}_h \in \mathfrak{M}(P)$ or $[\![b]\!]^{\mathfrak{M}}_h \in \mathfrak{M}(Q)$)

$\Leftrightarrow$ { interpretation of terms }

$\langle h(x), h(z) \rangle \in \mathfrak{M}(R)$ and ($\mathfrak{M}(a) \in \mathfrak{M}(P)$ or $\mathfrak{M}(b) \in \mathfrak{M}(Q)$)

$\Leftrightarrow$ { definition of h and $\mathfrak{M}$ }

$\langle 2, 2 \rangle \in \{\langle 0, 0 \rangle, \langle 2, 0 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle\}$ and ($0 \in \{0, 2\}$ or $1 \in \{1\}$)

$\Leftrightarrow$ { elementary set theory }

true and (true or true)

$\Leftrightarrow$

true

(b) $\mathfrak{M} \models \forall x(R(c,x) \to Q(x))[h]$

$\Leftrightarrow$ { definition of satisfaction for universal quantification }

$\mathfrak{M} \models (R(c,x) \to Q(x))[h[x/d]]$ for all $d \in \{0,1,2\}$

We claim that this is false. To show this, it suffices to demonstrate that $\mathfrak{M} \models (R(c,x) \to Q(x))[h[x/d]]$ is false for some $d \in \{0,1,2\}$. We shall do this for $d = 0$. We use k to abbreviate $h[x/0]$, so $k(x) = 0$. Now

$\mathfrak{M} \models (R(c,x) \to Q(x))[k]$

$\Leftrightarrow$ { definition of satisfaction for implication }

(not $\mathfrak{M} \models R(c,x)[k]$) or $\mathfrak{M} \models Q(x))[k]$

$\Leftrightarrow$ { definition of satisfaction for atomic formulae }

(not $\langle [\![c]\!]_k^{\mathfrak{M}}, [\![x]\!]_k^{\mathfrak{M}} \rangle \in \mathfrak{M}(R)$) or $[\![x]\!]_k^{\mathfrak{M}} \in \mathfrak{M}(Q)$

$\Leftrightarrow$ { interpretation of terms }

(not $\langle \mathfrak{M}(c), k(x) \rangle \in \mathfrak{M}(R)$) or $k(x) \in \mathfrak{M}(Q)$

$\Leftrightarrow$ { definition of $\mathfrak{M}$ and h }

(not $\langle 2,0 \rangle \in \{\langle 0,0 \rangle, \langle 2,0 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle\}$) or $0 \in \{1\}$

$\Leftrightarrow$ { elementary set theory }

(not true) or false

$\Leftrightarrow$

false

(c) $\mathfrak{M} \models \exists x \forall y R(x,y)[h]$

$\Leftrightarrow$ { definition of satisfaction for exiatential quantification }

there is a $d \in \{0,1,2\}$ such that $\mathfrak{M} \models \forall y R(x,y)[h[x/d]]$

$\Leftrightarrow$ { definition of satisfaction for universal quantification }

there is a $d \in \{0,1,2\}$ such that for all $e \in \{0,1,2\}$ $\mathfrak{M} \models R(x,y)[h[x/d,y/e]]$

$\Leftrightarrow$ { definition of satisfaction for atomic sentences }

there is a $d \in \{0,1,2\}$ such that for all $e \in \{0,1,2\}$ $\langle [\![x]\!]_{h[x/d,y/e]}^{\mathfrak{M}}, [\![y]\!]_{h[x/d,y/e]}^{\mathfrak{M}} \rangle \in \mathfrak{M}(R)$

$\Leftrightarrow$ { interpretation of terms }

there is a $d \in \{0,1,2\}$ such that for all $e \in \{0,1,2\}$ $\langle h[x/d,y/e](x), h[x/d,y/e](y) \rangle \in \mathfrak{M}(R)$

$\Leftrightarrow$

there is a $d \in \{0,1,2\}$ such that for all $e \in \{0,1,2\}$ $\langle d,e \rangle \in \mathfrak{M}(R)$

This last statement is true, for we have $\langle 2,0 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle \in \{\langle 0,0 \rangle, \langle 2,0 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle\}$.

7. (a) $\neg(A \leftrightarrow (B \lor \neg C))$

$\Leftrightarrow$

$\neg((A \to (B \lor \neg C)) \land ((B \lor \neg C) \to A))$

$\Leftrightarrow$

$\neg((\neg A \lor B \lor \neg C) \land (\neg(B \lor \neg C) \lor A))$

$\Leftrightarrow$

$\neg(\neg A \lor B \lor \neg C) \lor \neg(\neg(B \lor \neg C) \lor A)$

$\Leftrightarrow$

$(\neg\neg A \land \neg B \land \neg\neg C) \lor (\neg\neg(B \lor \neg C) \land \neg A)$

$\Leftrightarrow$

$(A \land \neg B \land C) \lor ((B \lor \neg C) \land \neg A)$

$\Leftrightarrow$

$(A \land \neg B \land C) \lor (B \land \neg A) \lor (\neg C \land \neg A)$

(b) $\neg(\forall x A(x) \to \forall x \exists y \forall z B(x, y, z))$

$\Leftrightarrow$ { rename x in the conclusion }

$\neg(\forall x A(x) \to \forall w \exists y \forall z B(w, y, z))$

$\Leftrightarrow$

$\neg(\neg\forall x A(x) \lor \forall w \exists y \forall z B(w, y, z))$

$\Leftrightarrow$

$\neg\neg\forall x A(x) \land \neg\forall w \exists y \forall z B(w, y, z)$

$\Leftrightarrow$

$\forall x A(x) \land \exists w \forall y \exists z \neg B(w, y, z)$

$\Leftrightarrow$

$\forall x \exists w \forall y \exists z (A(x) \land \neg B(w, y, z))$

Skolemize $\exists w$:

$\forall x \forall y \exists z (A(x) \land \neg B(f(x), y, z))$

Skolemize $\exists z$:

$\forall x \forall y (A(x) \land \neg B(f(x), y, g(x, y)))$

(c)

$((A$	$\land$	$B)$	$\to$	$\bot)$	$\land$	$((C$	$\land$	$A)$	$\to$	$D)$	$\land$	$(E$	$\to$	$C)$	$\land$	$(A$	$\to$	$E)$	$\land$	A
T	F	F	T	F	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T
1	5	5	5	5	5	3	3	1	4	4	5	2	3	3	5	1	2	2	5	1

In step 1, we assign T to A, because it is a conjunct of the Horn sentence.
In step 2, we observe that the premiss of conjunct $A \to E$ is true, so we assign T to E.
In step 3, we observe that the premiss of conjunct $E \to C$ is true, so we assign T to C.
In step 4, we observe that the premiss of conjunct $(C \land A) \to D$ is true, so we assign T to D.
In step 5, we observe that no new premises have become true, so we assign F to the remaining atom B. Now the Horn sentence evaluates to T.
We conclude that the Horn sentence is satisfiable.

8.

1. $\forall x(P \vee Q(x))$	
2. $\neg(P \vee \forall x Q(x))$	
3. $\boxed{a}$	
4. $P \vee Q(a)$	$\forall$ Elim: 1
5. P	
6. $P \vee \forall x Q(x)$	$\vee$ Intro: 5
7. $\bot$	$\bot$ Intro: 6, 2
8. $Q(a)$	$\bot$ Elim: 7
9. $Q(a)$	
10. $Q(a)$	Reit: 9
11. $Q(a)$	$\vee$ Elim: 4, 5–8, 9–10
12. $\forall x Q(x)$	$\forall$ Intro: 3–11
13. $P \vee \forall x Q(x)$	$\vee$ Intro: 12
14. $\bot$	$\bot$ Intro: 13, 2
15. $\neg\neg(P \vee \forall x Q(x))$	$\neg$ Intro: 2–14
16. $P \vee \forall x Q(x)$	$\neg$ Elim: 15